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CONSTRUCTION OF EFFECTIVE ALGORITHMS FOR SOLVING SYSTEMS OF ISOMORPHISM EQUATIONS FOR HYPERCOMPLEX NUMBER SYSTEMS USING EXPONENTIAL REPRESENTATION

The method of determining the isomorphism of hypercomplex number systems is investigated in the article by analyzing the representations of exponential functions in these systems. It is shown that such an approach significantly improves the efficiency of algorithms for solving systems of isomorphism equations.

Key words: *hypercomplex number systems, isomorphism, exponential functions, solving systems of equations, exponential representation.*

Problem statement. The quantity of calculations for solving specific scientific and technical problems largely depends on the organization of these calculations [1; 2]. At the same time, one of the most effective methods of organizing computations is the transition from the initial representation of information to a form in which operating with data becomes more productive.

The idea of a transition from one object space to another is fruitful in the field of hypercomplex calculus [3–6]. This is facilitated by the fact that among the set of hypercomplex number systems (HNS) of fixed dimension, there are subsets of systems that are isomorphic to each other. The two HNS are isomorphic if there is a one-to-one correspondence between them, that the image of the operation on the operands in one HNS is equal to the operation on the images of the same operands in another HNS [7]. This means that any computation can be done in any of the isomorphic HNS. The result will be the same, taking into account the translation of data and results from one system to another.

Two isomorphic HNS are similar to each other relative to their defining operations. But there may be differences, which are very interesting for developers of rational computing processes. The fact is that tables of multiplication of isomorphic HNS can vary greatly in the number of zero cells: in a strongly filled HNS there are few zeros, weakly filled – a lot. And, as a consequence, operating with hypercomplex numbers in a strongly-filled emergency is coextensive with the need to perform more operations on real numbers than in a weakly filled [8].

Experience in the development of mathematical models with the application of HNS shows the need to apply both types of HNS: highly filled – to identify models, weakly filled – to intensify the very process of modeling [9; 10; 14]. That is, for the successful use of HNS methods in mathematical modeling, it is necessary to have a set of pairs of isomorphic HNS of different dimensions and types.

A significant obstacle here is the difficulty of establishing isomorphism (or lack thereof) of two HNS. Proceeding from this, the goal of this work is determined.

Purpose of the article. The creation of such algorithms for solving systems of quadratic isomorphism equations for the pair HNS, which would greatly simplify their solution. The goal is achieved by using representations of exponential functions in these HNS.

The equations system of HNS isomorphism. Hypercomplex number systems Γ_1 and Γ_2 are called isomorphic ($\Gamma_1 \simeq \Gamma_2$) if there exists a one-to-one mapping L of space Γ_1 to space Γ_2 such that the following properties hold:

$$L(a+b) = L(a) + L(b), \qquad (1)$$

$$L(a \times b) = L(a) \times L(b), \qquad (2)$$

where $a,b \in \Gamma_1$, $L(a)$, $L(b) \in \Gamma_2$.

The operations of multiplication in the left and right parts of (2) differ from each other in accordance with the structure constants Γ_1 and Γ_2.

It follows from (2) that Γ_1 and Γ_2 are linear spaces with bases $e = \{ e_1, e_2, ..., e_n \}$ and $f = \{ f_1, f_2, ..., f_n \}$, respectively, and therefore one can establish a one-to-one linear correspondence between them with the real matrix A [7; 8]:

$$\begin{aligned} e_1 &= \alpha_{11} f_1 + \alpha_{12} f_2 + ... + \alpha_{1n} f_n \\ e_2 &= \alpha_{21} f_1 + \alpha_{22} f_2 + ... + \alpha_{2n} f_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_n &= \alpha_{n1} f_1 + \alpha_{n2} f_2 + ... + \alpha_{nn} f_n . \end{aligned} \quad (3)$$

In this case, the determinant of the matrix A is different from zero $\|A\| \uparrow 0$, since the transformation (3) must have an inverse transformation A^{-1}.

As follows from the theory of linear spaces, for each pair of linearly independent bases one can find a one-to-one linear transformation (3), which takes one basis to another and vice versa. But the fulfillment simultaneously with the requirement (1) and requirement (2), which reduces to a system of nonlinear algebraic equations, is not always possible.

If $a = \sum_{i=1}^{n} a_i e_i$, $b = \sum_{i=1}^{n} b_i e_i$, $ab = \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} a_i b_j \gamma_{ij}^k e_k$,

then $L(ab) = L\left(\sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} a_i b_j \gamma_{ij}^k e_k \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{ij}^k a_i b_j L(e_k)$.

But from (3) $L(e_k) = \sum_{s=1}^{n} \alpha_{ks} f_s$.

Therefore

$$L(ab) = \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{s=1}^{n} \alpha_{ks} \gamma_{ij}^k a_i b_j f_s . \quad (4)$$

On the other side

$$\begin{aligned} L(a) \times L(b) &= \sum_{i=1}^{n} a_i L(e_i^1) \sum_{j=1}^{n} b_j L(e_j) = \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{ik} a_i f_k \sum_{j=1}^{n} \sum_{s=1}^{n} \alpha_{js} b_j f_s = \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{s=1}^{n} \sum_{y=1}^{n} \alpha_{ik} \alpha_{js} \gamma_{ks}^y a_i b_j f_y . \end{aligned} \quad (5)$$

Equating the expressions in the right-hand sides of equations (4) and (5) for identical ones $a_i b_j f_y$, we obtain nonlinear algebraic equations from n^2 unknown α_{ij}.

$$\sum_{k=1}^{n} \sum_{s=1}^{n} \alpha_{ks} \gamma_{ij}^k = \sum_{k=1}^{n} \sum_{s=1}^{n} \sum_{r=1}^{n} \alpha_{ik} \alpha_{js} \gamma_{ks}^r ; \quad i,j,k \in 1,...,n . \quad (6)$$

This system is overridden. It always has a trivial solution $\alpha_{ij} = 0; \quad i,j = 1,...,n$.

But non-trivial real solutions can not exist if the condition $\|A\| \uparrow 0$ is satisfied. Therefore, if there exists at least one non-trivial real solution, then these two HNS Γ_1 and Γ_2 are isomorphic, if there are no such solutions, then they are not isomorphic.

The solution of such systems causes considerable difficulties even when using such powerful systems of analytical computations as MAPLE, MATHEMATICA etc. When using the MAPLE system, systems of equations for n = 3 are successfully solved. But even at n = 4, the solution time increases to many hours and in many cases it was not possible to obtain a solution at all.

In view of the foregoing, studies in the direction of developing such methods for establishing isomorphism between HNS that do not require the solution of quadratic systems of type (6) are of great relevance, or at least greatly simplify their solution.

As our studies have shown, significant progress in this direction can be achieved by applying representations of exponential functions in HNS.

Representations of exponential functions in HNS. We consider the main features of one of the universal methods for constructing an exponential with the help of an associated system of linear differential equations [4, p. 11–13].

Let and hypercomplex numbers:

$$X = \sum_{i=1}^{n} x_i e_i ; \; M = \sum_{i=1}^{n} m_i e_i , \quad (7)$$

where $\overline{X} = (x_1, ..., x_n)^T$, $\overline{M} = (m_1, ..., m_n)^T$ – vector columns composed of components of hypercomplex numbers.

The representation of the exponent in HNS $\Gamma(e,n)$ from the number $M \in \Gamma(e,n)$ that we denote is a particular solution of the ordinary hypercomplex linear differential equation

$$\dot{X} = MX , \quad (8)$$

with the initial condition $Exp(0) = \varepsilon$, where $\Gamma(e,n)$ – HNS of dimension n with a basis e and a unit element ε [4]. The differentiation in (8) is assumed by the scalar argument.

To construct a solution of the hypercomplex linear differential equation (8), it must be represented in vector-matrix form. In this case $\dot{\overline{X}} = (\dot{x}_1, ..., \dot{x}_n)^T$ and the column vector obtained from the hypercomplex number can be represented in the form of a matrix product of some matrix with dimensions whose elements are linear combinations of the components of the hypercomplex number by the column vector , that is .

Then the hypercomplex equation (8) becomes a system of n equations, which is called the associated system of linear differential equations

$\dot{\overline{X}} = œ \overline{X}$. (9)

It is necessary to find the characteristic numbers $\lambda_1, ..., \lambda_n$ of the matrix œ, that is, to solve the characteristic equation $œ - \lambda E = 0$. Thus, the characteristic numbers $\lambda_1, ..., \lambda_n$ will depend on the hypercomplex number M.

After this, it is necessary to construct a general solution, depending on the n^2 arbitrary constants, of which $n^2 - n$ are linearly dependent on n free variables. To obtain these linear dependences, it is necessary to solve a system of linear equations [1; 3]. After this, one can obtain general solutions (9), which depend on n arbitrary constants $\bar{X}(t, C_1, ..., C_n)$. The values of arbitrary constants are established using the initial condition $Exp(0) = \varepsilon$. The components of the vector-column $\bar{X}$ –solution and are components of the exponent of the hypercomplex number M

$$Exp(M) = \sum_{i=1}^{n} \bar{x}_i e_i . \qquad (10)$$

The method of constructing representations of an exponential from a hypercomplex variable with the help of an associated system of linear differential equations is fairly easy to formalize for the construction of algorithms and programs in systems of symbolic computations.

Normalized form of the exponential representation. In the general case, the set of roots $\lambda_1, ..., \lambda_n$ of the characteristic equation $œ - \lambda E = 0$ consists of n roots and can be divided into the following subsets:

1. A subset of single real roots $\lambda_i \in R$.

In the exponential representation, they correspond to terms of the form

$$x_i = \overline{x_i} \cdot e_i = C_i e^{\lambda_i} e_i .$$

2. A subset of conjugate pairs of complex roots $\lambda_i, \lambda_{i+1} = \overline{\lambda_i} \in C$.

Usually, when solving systems of linear differential equations for a pair of complex conjugate roots, the particular solution is taken in the form $x = e^{\mathrm{Re}(\lambda t)}\left(C_1 \cos(\mathrm{Im}(\lambda) t) + C_2 \sin(\mathrm{Im}(\lambda) t)\right)$. In this paper, Euler's formula will not be used to write the solution, as will the representation of the real exponential in terms of hyperbolic functions $e^{\phi} = ch\phi + sh\phi$, since this greatly complicates the structure of the representation formula and makes it difficult to analyze it. Instead, for a pair of complex-conjugate roots, the components of the representation are written as two

$$x_i = \overline{x_i} \cdot e_i = C_i e^{\lambda_i} e_i ; \qquad x_{i+1} = \overline{x}_{i+1} \cdot e_{i+1} = \overline{C}_i e^{\overline{\lambda_i}} e_{i+1} ,$$

but arbitrary constants are no longer real, but complex.

3. A subset of real multiple roots.

Suppose that the multiplicity of one of the collections of real multiple roots is s

$$\lambda_{i+1} = \lambda_{i+2} = ...\lambda_{i+s}$$

Then, as follows from the theory of linear differential equations, this set of roots will correspond to s components of a general solution of the form

$$x_{i+j} = \overline{x}_{i+j} e_{i+j} = \left(P_0^j + P_1^j + ... + P_s^j\right) e^{\lambda_{i+j}} e_{i+j} ; \; j = 1, ..., s ,$$

where P_k^j – polynomial of degree k in the variables $m_1, ..., m_n$. The form of these polynomials is determined from the defining equation of the associated system of linear differential equations.

4. A subset of multiple pairs of complex-conjugate roots.

Suppose that the multiplicity of one of the sets of multiple pairs of complex conjugate roots is equal to s. Then all in this set will be the $2s$ roots

$$\lambda_{i+1} = \lambda_{i+3} = ... = \lambda_{i+2s-1} ; \qquad \lambda_{i+2} = ... = \lambda_{i+2s} = \overline{\lambda}_{i+1} .$$

And this set of roots will correspond to $2s$ components of the general solution of the form

$$x_{i+j} = \overline{x}_{i+j} e_{i+j} = \left(P_0^j + P_1^j + ... + P_s^j\right) e^{\lambda_{i+j}} e_{i+j} ;$$
$$x_{i+j+1} = \overline{x}_{i+j+1} e_{i+j+1} = \left(\overline{P}_0^j + \overline{P}_1^j + ... + \overline{P}_s^j\right) e^{\overline{\lambda}_{i+j}} e_{i+j+1} ;$$
$$j = 1, 3, ..., 2s - 1 .$$

Here there will already be polynomials with complex coefficients.

Thus, the representation of the exponential will represent the sum of the n summands, each of which is a monomial, for which in the first two cases there are three factors: a real or complex arbitrary constant, an exponential of the real or complex characteristic root, and a basic element. In the third and fourth cases there are four factors. To the three previous factors we add a polynomial of the $(s-1)$ power with real or complex variables. Such a form of the representation of an exponential will be called the normalized form of the representation.

The action of the isomorphism operator on the representation of the exponent. The isomorphism of two HNS means the existence of such a linear transformation of bases whose determinant is not equal to zero, that for operations of addition and multiplication the image of the result of these operations is equal to the result of the operation on operands. Therefore, any expression with a finite number of hypercomplex operations is transformed by the same linear transformation.

The representation of an exponential in terms of a power series contains a countable number of operations. However, in this case, as will be shown below, an isomorphic transformation of the representation of an exponent from a number in one HNS will lead to a representation of the exponent from the image of this number in another

Let two isomorphic HNS $\Gamma_1(e, n) \simeq \Gamma_2(f, n)$ and a linear isomorphic transformation L

$$\Gamma_1(e, n) \overset{L}{\simeq} \Gamma_2(f, n) , \qquad (11)$$

$$L: e_k = \sum_{j=1}^{n} l_{kj} f_j ; \quad k = 1, ..., n . \qquad (12)$$

The number $X = \sum_{i=1}^{n} x_i e_i \in \Gamma_1(e, n)$ in the transition to the system $\Gamma_2(f, n)$ with the L isomorphism is transformed as follows:

$$X = x_1 e_1 + x_2 e_2 + ... + x_n e_n \Leftrightarrow x_1\left(l_{11} f_1 + l_{12} f_2 + ... + l_{1n} f_n\right) +$$
$$+ x_2\left(l_{21} f_1 + l_{22} f_2 + ... + l_{2n} f_n\right) + ... + x_n\left(l_{n1} f_1 + l_{n2} f_2 + ... + l_{nn} f_n\right) = \qquad (13)$$
$$= \left(x_1 l_{11} + x_2 l_{21} + ... + x_n l_{n1}\right) f_1 + \left(x_1 l_{12} + x_2 l_{22} + ... + x_n l_{2n}\right) f_2 +$$
$$+ ... + \left(x_1 l_{1n} + x_2 l_{2n} + ... + x_n l_{nn}\right) f_n = y_1 f_1 + ... + y_n f_n \in \text{“}_2(f, n),$$

where

$$y_i = x_1 l_{1i} + x_2 l_{2i} + ... + x_n l_{ni} . \quad (14)$$

That

$$\bar{Y} = L^T \bar{X} , \quad (15)$$

the components of the hypercomplex number $Y \in \Gamma_2(f,n)$ (column vector $\bar{Y}$) are obtained by multiplying the column vector $\bar{X}$ from the left by the trans-rotated matrix of the isomorphic transformation operator L^T .

If we apply the linear transformation of the isomorphism L to the exponent of the hypercomplex number $X = \sum_{i=1}^{n} x_i e_i \in \Gamma_1(e,n)$, we get the exponential of the hypercomplex number $Y \in \Gamma_2(f,n)$, which is the image of the number X :

$$Exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \overset{L}{\Leftrightarrow} \sum_{k=0}^{\infty} \frac{Y^k}{k!} = Exp(Y) \in \Gamma_2(f,n) . \quad (16)$$

Hence, subjecting an isomorphic transformation to an exponent in one HNS, one can obtain an exponent in isomorphic HNS from image numbers. The same can be said about representations of exponentials, since their construction with respect to a power series gives a unique representation.

We state the main result: if there are two isomorphic systems (11) and an isomorphism operator (12), then an isomorphic transformation of the representation of an exponent in one of the HNS is a representation of the exponent in another HNS.

The set of roots of the characteristic equation and the isomorphism of HNS. Let us consider the case when a hypercomplex number system $\Gamma_1(e,n)$ is a direct sum of number systems Γ_{1i}

$$\Gamma_1 = \bigoplus_{i=1}^{k} \Gamma_{1i} . \quad (17)$$

As was shown above, the normalized form of the representation of its exponential consists of normalized forms of representations of the exponent of each of the incoming subsystems. That is, the number of summands is equal to the number of roots of the characteristic equation, which in turn is equal to the dimension of the entire HNS. Each term is determined by one of the roots of the characteristic equation.

We proceed by a linear transformation of a basis e from a system $\Gamma_1(e,n)$ to a system $\Gamma_2(f,n)$ isomorphic to it. Let us consider how the roots of the characteristic equation, which appear in the summands of the exponentials of the system $\Gamma_1(e,n)$, change in this case. Since these roots are functions of the components of a number $\overline{M}$, they will vary according to (15), that is, multiplying the column vector $\overline{M}$ from the left by the transposed matrix of the isomorphic transformation operator L^T . Hence, the roots of the characteristic equation are linearly transformed. And this means that their type does not change: different material roots go into different real, different complex into different complex, identical roots – into the same, the real can not be transformed into complex ones and vice versa. The characteristic equation $œ - \lambda E = 0$ can be represented in the form

$$(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) = 0 , \quad (18)$$

where λ_i , $i = 1,...,n$ – roots of the characteristic equation. Since they depend on the components $\overline{M}$, the linear transformation of them does not change the type. And this means that the normal form of the exponent of the system $\Gamma_2(f,n)$ has the same structure as the exponent in the system $\Gamma_1(e,n)$.

If the basis of the system $\Gamma_1(e,n)$ is transformed by another linear transformation, then we obtain a system $\Gamma_3(g,n)$ isomorphic $\Gamma_1(e,n)$:

$$\Gamma_3(g,n) \simeq \Gamma_1(e,n)$$

and, under condition the transitivity of the isomorphism relation, we obtain

$$\Gamma_3(g,n) \simeq \Gamma_2(f,n) .$$

Thus, by transforming by any possible non-degenerate linear transformations any basis to which a fixed set of roots of the characteristic equation corresponds, one can obtain the whole class of isomorphisms. That is, one and only one set of roots of the characteristic equation will correspond to this class of isomorphisms.

Unfortunately, the converse is not true. As follows from [8], different nonisomorphic HNS can correspond to the same set of roots. This can happen when the roots of the characteristic equation have multiple real or (or) complex roots of multiplicity greater than 2. Several classes of isomorphisms of indecomposable HNS correspond to roots of this multiplicity, and only one isomorphism class of the system of dual numbers D with the table multiplication:

D	e_1	e_2
e_1	e_1	e_2
e_2	e_1	0

The multiplicities 3 correspond to two classes of isomorphisms. Tables of multiplication of representatives of classes are given below.

In both HNS, the characteristic equations have threefold roots .

The multiplicities 4 correspond to 6 isomorphism classes. Tables of multiplication of representatives of classes are given below.

Γ_{31}	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_1	0	0
e_3	e_1	0	0

Γ_{32}	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_3	0
e_3	e_3	0	0

Γ_{31}	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	$-e_4$	$-e_3$
e_3	e_3	e_4	0	0
e_4	e_4	$-e_3$	0	0

Γ_{42}	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	0	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

Γ_{43}	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_3	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

Γ_{44}	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

Γ_{45}	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	0	0
e_3	e_3	0	$-e_4$	0
e_4	e_4	0	0	0

Γ_{46}	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	e_4	0
e_3	e_3	e_4	0	0
e_4	e_4	0	0	0

In HNS, the characteristic equation has a double pair of complex conjugate roots: $\lambda_{1,2}=m_1 \pm im_2$; $\lambda_{3,4}=m_1 \pm im_2$ The remaining HNS have real fourfold roots $\lambda_{1,2,3,4}=m_1$. Therefore, judging by the characteristic roots, it can always be asserted that the system is not isomorphic to any of the other systems and vice versa. But about isomorphism in the totality of systems Γ_{42}, Γ_{43}, Γ_{44}, Γ_{45}, Γ_{46} only according to the characteristic roots nothing can be said. Their non-isomorphism was established by directly solving the systems of equations (6), which took a very long time.

Practical example. As an example, consider a pair of HNS: a bicomplex system $C \oplus C(e,4)$ and a system of quadriplex numbers whose multiplication tables are given below, $K(f,4)$ and we solve the question of their isomorphism.

$C \oplus C$	e_1	e_2	e_3	e_4
e_1	e_1	e_2	0	0
e_2	e_2	$-e_1$	0	0
e_3	0	0	e_3	e_4
e_4	0	0	e_4	$-e_3$

K	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	e_3
e_3	e_3	e_4	$-e_1$	$-e_2$
e_4				e_1

We assume that the isomorphism operator has the most general form

$$L: \begin{cases} e_1 = x_{11} f_1 + x_{12} f_2 + x_{13} f_3 + x_{14} f_4; \\ e_3 = x_{31} f_1 + x_{32} f_2 + x_{33} f_3 + x_{34} f_4; \\ e_2 = x_{21} f_1 + x_{22} f_2 + x_{23} f_3 + x_{24} f_4; \\ e_4 = x_{41} f_1 + x_{42} f_2 + x_{43} f_3 + x_{44} f_4 \end{cases} \qquad (19)$$

Since the unit elements of these systems are correspondingly $\varepsilon_{C \oplus C}=e_1+e_3$, $\varepsilon_K=f_1$, then the first equation of system (19) could be taken $e_1+e_3=f_1$ in such a way that it would somewhat simplify the problem. However, to demonstrate the universality of the method, we will not use this simplifying preliminary information. To solve the problem by the traditional method, it is necessary to compile a system (6), which in this case will consist of 24 quadratic equations [1].

$$\begin{aligned} & x_{21}^2 - x_{22}^2 = -1; \;\; 2x_{21}x_{22} = 0; \\ & x_{23}^2 - x_{24}^2 = -1; \;\; 2x_{23}x_{24} = 0; \\ & x_{31}^2 - x_{32}^2 = -1; \;\; 2x_{31}x_{32} = 0; \\ & x_{33}^2 - x_{34}^2 = -1; \;\; 2x_{33}x_{34} = 0; \\ & x_{41}^2 - x_{42}^2 = 1; \; 2\alpha_{41}\alpha_{42} = 0; \\ & x_{43}^2 - x_{44}^2 = 1; \;\; 2x_{43}x_{44} = 0; \\ & x_{21}x_{31} - x_{22}x_{32} = x_{41}; \;\; x_{22}x_{31} + x_{21}x_{32} = x_{42}; \\ & x_{23}x_{33} - x_{24}x_{34} = x_{43}; \;\; x_{23}x_{34} + x_{24}x_{33} = x_{44}; \\ & x_{21}x_{41} - x_{22}x_{42} = -x_{31}; \;\; x_{22}x_{41} + x_{21}x_{42} = -x_{32}; \\ & x_{23}x_{43} - x_{24}x_{44} = -x_{33}; \;\; x_{23}x_{44} + x_{24}x_{43} = -x_{34}; \\ & x_{31}x_{41} - x_{32}x_{42} = -x_{21}; \;\; x_{32}x_{41} + x_{31}x_{42} = -x_{22}; \\ & x_{33}x_{43} - x_{32}x_{44} = -x_{23}; \;\; x_{33}x_{44} + x_{34}x_{43} = -x_{24}. \end{aligned} \qquad (20)$$

It should be noted that the relationship between the unitary elements is taken into account. Otherwise, the number of equations in the system (20) would increase to 40. As we see, the equations of the quadratic system (20) have a complicated structure, and a large combinatoriality arises when solving. As the solution shows with the help of the system of symbolic computation MAPLE, it has 8 solutions that satisfy the condition $\|A\| \uparrow 0$. Therefore, we can conclude that the systems $C \oplus C(e,4)$ and $K(f,4)$ are also isomorphic.

We give one of the non-degenerate solutions of system (20):

$$\begin{aligned} & x_{11} = 1; \quad x_{12} = 0; \quad x_{13} = 1; \\ & x_{14} = 0; \quad x_{21} = 0; \quad x_{22} = -1; \quad x_{23} = 0; \quad x_{24} = 1; \\ & x_{31} = 0; \quad x_{32} = -1; \quad x_{33} = 0; \\ & x_{34} = -1; \quad x_{41} = -1; \quad x_{42} = 0; \quad x_{43} = 1; \quad x_{44} = 0. \end{aligned} \qquad (21)$$

The direct and inverse isomorphism operators take the corresponding form:

$$L: \begin{cases} f_1 = e_1 + e_3; \quad f_3 = -e_2 - e_4; \\ f_2 = -e_2 + e_4; \, f_4 = -e_1 + e_3 \end{cases};$$

$$L^{-1}: \begin{cases} e_1 = \frac{1}{2} f_1 - \frac{1}{2} f_4 \;; e_2 = -\frac{1}{2} f_2 - \frac{1}{2} f_3; \\ e_3 = \frac{1}{2} f_1 + \frac{1}{2} f_4; e_4 = \frac{1}{2} f_2 - \frac{1}{2} f_3 \end{cases} \qquad (22)$$

We solve the same problem by means of representations of exponentials. Let $M = \sum_{j=1}^{4} m_j e_j \in C \oplus C$, $N = \sum_{j=1}^{4} n_j f_j \in K$. Then, as shown in [1]:

$$\begin{aligned} Exp(N) = \frac{1}{2} e^{n_1} [& \left(e^{-n_4} \cos(n_2 + n_3) + e^{n_4} \cos(-n_2 + n_3)\right) \\ & f_1 + \left(e^{-n_4} \sin(n_2 + n_3) - e^{n_4} \sin(-n_2 + n_3)\right) f_2 + \\ & + \left(e^{-n_4} \sin(n_2 + n_3) + e^{n_4} \sin(-n_2 + n_3)\right) \\ & f_3 + \left(-e^{-n_4} \cos(n_2 + n_3) + e^{n_4} \cos(-n_2 + n_3)\right) f_4]. \end{aligned}$$

$$Exp(M) = e^{m_1} \left(\cos m_2 \cdot e_1 + \sin m_2 \cdot e_2\right) + e^{m_3} \left(\cos m_4 \cdot e_3 + \sin m_4 \cdot e_4\right)$$

We translate these representations into a normal form, for which instead of trigonometric functions one must substitute their expressions in terms of exponentials with imaginary exponents by the Euler formula, and make a rearrangement of the terms. As a result, we get the same expressions, but with different constants and characteristic roots

$$Exp(K) = C_1 e^{\lambda_1} + \overline{C_1} e^{\overline{\lambda_1}} + C_2 e^{\lambda_2} + \overline{C_2} e^{\overline{\lambda_2}}, \qquad (23)$$

where for system $C \oplus C(e,4)$

$$K = N,\ C_1 = \frac{1}{2}(e_1 - ie_2), C_2 = \frac{1}{2}(e_3 - ie_4),$$

$$\lambda_1 = \mu_1 = m_1 + im_2,\ \lambda_2 = \mu_2 = m_3 + im_4,$$

and for system $K(f,4)$

$$K = M,\ C_1 = \frac{1}{4}(f_1 - if_2 - if_3 - f_4),$$

$$C_2 = \frac{1}{4}(f_1 + if_2 - if_3 + f_4), \qquad (24)$$

$$\lambda_1 = \nu_1 = m_1 - m_4 + i(m_3 + m_2),$$

$$\lambda_2 = \nu_2 = m_1 + m_4 + i(m_3 - m_2).$$

Already from the fact that the exponent representations in both HNS have one type of set of roots of characteristic equations: two different pairs of complex roots, allow us to conclude about the isomorphism of systems $C \oplus C(e,4)$ and $K(f,4)$. And this despite the fact that there is no need to solve a cumbersome quadratic system (20).

This trip allows one to obtain an explicit form of linear transformation (19). Let us construct the law of the transformation of numbers under an isomorphic transition. From relation

$$M = \sum_{j=1}^{4} m_j e_j \overset{L}{\Leftrightarrow} \sum_{j=1}^{4} n_j f_j = N \qquad (25)$$

and result

$$n_i = \sum_{j=1}^{4} m_j x_{ij}, \qquad i = 1,\ldots,4. \qquad (26)$$

The desired conversion L must translate the representation in the system $C \oplus C(e,4)$ into a representation in the system $K(f,4)$:

$$\begin{aligned} &\frac{1}{2}[(e_1 - ie_2)e^{\mu_1} + (e_1 + ie_2)e^{\overline{\mu_1}} + \\ &+ (e_3 - ie_4)e^{\mu_2} + \frac{1}{2}(e_3 + ie_4)e^{\overline{\mu_2}}] \overset{L}{\Leftrightarrow} \\ &\overset{L}{\Leftrightarrow} \frac{1}{4}[(f_1 - if_2 - if_3 - f_4)e^{\nu_1} + (f_1 + if_2 + if_3 - f_4)e^{\overline{\nu_1}} + \\ &+ (f_1 + if_2 - if_3 + f_4)e^{\nu_2} + (f_1 - if_2 + if_3 + f_4)e^{\overline{\nu_2}}]. \end{aligned} \qquad (27)$$

We substitute the transformation (19) into the left-hand side of (27):

$$\begin{aligned} &\frac{1}{2}(x_{11}f_1 + x_{12}f_2 + x_{13}f_3 + x_{14}f_4 - i(x_{21}f_1 + x_{22}f_2 + x_{23}f_3 + x_{24}f_4))e^{\mu_1} + \\ &+ \frac{1}{2}(x_{11}f_1 + x_{12}f_2 + x_{13}f_3 + x_{14}f_4 + i(x_{21}f_1 + x_{22}f_2 + x_{23}f_3 + x_{24}f_4))e^{\overline{\mu_1}} + \\ &+ \frac{1}{2}(x_{31}f_1 + x_{32}f_2 + x_{33}f_3 + x_{34}f_4 - i(x_{41}f_1 + x_{42}f_2 + x_{43}f_3 + x_{44}f_4))e^{\mu_2} + \\ &+ \frac{1}{2}(x_{31}f_1 + x_{32}f_2 + x_{33}f_3 + x_{34}f_4 + i(x_{41}f_1 + x_{42}f_2 + x_{43}f_3 + x_{44}f_4))e^{\overline{\mu_2}} \overset{L}{\Leftrightarrow} \\ &\overset{L}{\Leftrightarrow} \frac{1}{4}(f_1 - if_2 - if_3 - f_4)e^{\nu_1} + \frac{1}{4}(f_1 + if_2 + if_3 - f_4)e^{\overline{\nu_1}} + \\ &+ \frac{1}{4}(f_1 + if_2 - if_3 + f_4)e^{\nu_2} + \frac{1}{4}(f_1 - if_2 + if_3 + f_4)e^{\overline{\nu_2}}. \end{aligned} \qquad (28)$$

Since the correspondence (28) must be satisfied for any values of the roots $\propto$ and ν, then the coefficients x_{ij} can be found by the method of undetermined coefficients with respect to the exponentials. Here, you can combine these coefficients in different ways. In this case, transformations of various types can be obtained, including degenerate ones (which do not satisfy the condition $\|A\| \uparrow 0$). Degenerate transformations indicate an unacceptable way of combining. Let's choose a way of combining the coefficients:

$$\mu_1 \Leftrightarrow \nu_1;\ \overline{\mu_1} \Leftrightarrow \overline{\nu_1};\ \mu_2 \Leftrightarrow \nu_2;\ \overline{\mu_2} \Leftrightarrow \overline{\nu_2}. \qquad (29)$$

This correspondence will give a system of 4 equations that, by the method of undetermined coefficients relative to the basis elements and the imaginary unit, will yield a system of 16 very simple linear equations

$$\begin{array}{lll} 2x_{11} - 2ix_{21} = 1 & 2x_{12} - 2ix_{22} = -i & 2x_{13} - 2ix_{23} = -i \\ 2x_{11} + 2ix_{21} = 1 & 2x_{12} + 2ix_{22} = i & 2x_{13} + 2ix_{23} = i \\ 2x_{31} - 2ix_{41} = 1 & 2x_{32} - 2ix_{42} = i & 2x_{33} - 2ix_{43} = -i \\ 2x_{31} + 2ix_{41} = 1 & 2x_{32} + 2ix_{42} = -i & 2x_{33} + 2ix_{43} = i \end{array}$$

$$\begin{array}{l} 2x_{14} - 2ix_{24} = -1 \\ 2x_{14} + 2ix_{24} = 1 \\ 2x_{34} - 2ix_{44} = 1 \\ 2x_{34} + 2ix_{44} = 1 \end{array}, \qquad (30)$$

the solution of which has the form

$$\begin{aligned} &x_{11} = \frac{1}{2},\ x_{12} = 0,\ x_{13} = 0,\ x_{14} = \\ &= -\frac{1}{2},\ x_{21} = 0,\ x_{22} = \frac{1}{2},\ x_{23} = \frac{1}{2},\ x_{24} = 0 \\ &x_{31} = \frac{1}{2},\ x_{32} = 0,\ x_{33} = 0,\ x_{34} = \frac{1}{2}, \\ &x_{41} = 0,\ x_{42} = -\frac{1}{2},\ x_{43} = \frac{1}{2},\ x_{44} = 0, \end{aligned} \qquad (31)$$

and the isomorphism operator

$$L: \begin{cases} e_1 = \frac{1}{2}(f_1 - f_4); & e_2 = \frac{1}{2}(f_2 + f_3); \\ e_3 = \frac{1}{2}(f_1 + f_4); & e_4 = \frac{1}{2}(-f_2 + f_3). \end{cases}, \qquad (32)$$

which differs from (22). This is because the isomorphism operator can not be unique. The implementation of a particular type of operator depends on the method of combining the roots (29) in the compilation of the system (30). In any case, as is easily seen directly, the resulting operator takes the system of quadriplex numbers $K(f,4)$ to a system of bicomplex numbers $C \oplus C(e,4)$.

So, for example,

$$e_3e_4 = \frac{1}{4}(-f_2 + f_3 + f_3 - f_2) = \frac{1}{2}(-f_2 + f_3) = e_4$$

that corresponds to the multiplication table of the Keli system $C \oplus C(e,4)$. It is also easy to verify directly with the aid of (26) that the operator (32) satisfies (29).

Conclusions. Thus, the method of investigating the isomorphism of hypercomplex number systems by analyzing the representations of exponential functions in these systems with single roots of the characteristic equation of HNS makes it possible to significantly improve the efficiency of algorithms for solving systems of isomorphism equations by eliminating the need for solving cumbersome systems of quadratic equations. At the same time, it should be noted that the presence of multiple roots of the characteristic equations requires additional studies, which will be performed in the future.

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ПОСТРОЕНИЕ ЭФФЕКТИВНЫХ АЛГОРИТМОВ РЕШЕНИЯ СИСТЕМ УРАВНЕНИЙ ИЗОМОРФИЗМА ГИПЕРКОМПЛЕКСНЫХ ЧИСЛОВЫХ СИСТЕМ С ИСПОЛЬЗОВАНИЕМ ЭКСПОНЕНЦИАЛЬНОГО ПРЕДСТАВЛЕНИЯ

В статье исследуется метод определения изоморфизма гиперкомплексных числовых систем путем анализа представления экспоненциальных функций в этих системах. Показано, что такой подход значительно повышает эффективность алгоритмов решения систем уравнений для определения изоморфизма.

Ключевые слова: *гиперкомплексные числовые системы, изоморфизм, экспоненциальные функции, решения систем уравнений, экспоненциальное представление.*

ПОБУДОВА ЕФЕКТИВНИХ АЛГОРИТМІВ РОЗВ'ЯЗАННЯ СИСТЕМ РІВНЯНЬ ІЗОМОРФІЗМУ ГІПЕРКОМПЛЕКСНИХ ЧИСЛОВИХ СИСТЕМ ІЗ ВИКОРИСТАННЯМ ЕКСПОНЕНЦІАЛЬНОГО ПРЕДСТАВЛЕННЯ

У статті досліджується метод визначення ізоморфізму гіперкомплексних цифрових систем за допомогою аналізу представлення експоненціальних функцій у цих системах. Показано, що такий підхід значно підвищує ефективність алгоритмів розв'язання систем рівнянь для визначення ізоморфізму.

Ключові слова: *гіперкомплексні числові системи, ізоморфізм, експоненціальні функції, розв'язання систем рівнянь, експоненціальне представлення.*